

# METRICAL CHARACTERIZATION OF SUPER-REFLEXIVITY AND LINEAR TYPE OF BANACH SPACES

FLORENT BAUDIER<sup>†</sup>

**ABSTRACT.** We prove that a Banach space  $X$  is not super-reflexive if and only if the hyperbolic infinite tree embeds metrically into  $X$ . We improve one implication of J.Bourgain's result who gave a metrical characterization of super-reflexivity in Banach spaces in terms of uniforms embeddings of the finite trees. A characterization of the linear type for Banach spaces is given using the embedding of the infinite tree equipped with the metrics  $d_p$  induced by the  $\ell_p$  norms.

## 1. INTRODUCTION AND NOTATION

We fix some notation and recall basic results.

Let  $(M, d)$  and  $(N, \delta)$  be two metric spaces and an injective map  $f : M \rightarrow N$ . Following [11], we define the *distortion* of  $f$  to be

$$\text{dist}(f) := \|f\|_{Lip} \|f^{-1}\|_{Lip} = \sup_{x \neq y \in M} \frac{\delta(f(x), f(y))}{d(x, y)} \cdot \sup_{x \neq y \in M} \frac{d(x, y)}{\delta(f(x), f(y))}.$$

If  $\text{dist}(f)$  is finite, we say that  $f$  is a metric embedding, or simply an embedding of  $M$  into  $N$ .

And if there exists an embedding  $f$  from  $M$  into  $N$ , with  $\text{dist}(f) \leq C$ , we use the notation  $M \xrightarrow{C} N$ .

Denote  $\Omega_0 = \{\emptyset\}$ , the root of the tree. Let  $\Omega_n = \{-1, 1\}^n$ ,  $T_n = \bigcup_{i=0}^n \Omega_i$  and  $T = \bigcup_{n=0}^{\infty} T_n$ . Thus  $T_n$  is the finite tree with  $n$  levels and  $T$  the infinite tree.

For  $\varepsilon, \varepsilon' \in T$ , we note  $\varepsilon \leq \varepsilon'$  if  $\varepsilon'$  is an extension of  $\varepsilon$ .

Denote  $|\varepsilon|$  the length of  $\varepsilon$ ; i.e the numbers of nodes of  $\varepsilon$ . We define the hyperbolic distance between  $\varepsilon$  and  $\varepsilon'$  by  $\rho(\varepsilon, \varepsilon') = |\varepsilon| + |\varepsilon'| - 2|\delta|$ , where  $\delta$  is the greatest common ancestor of  $\varepsilon$  and  $\varepsilon'$ . The metric on  $T_n$  is the restriction of  $\rho$ .

For a Banach space  $X$ , we denote  $B_X$  its closed unit ball, and  $X^*$  its dual space.

$T$  embeds isometrically into  $\ell_1(\mathbb{N})$  in a trivial way. Actually, let  $(e_\varepsilon)_{\varepsilon \in T}$  be the canonical basis of  $\ell_1(T)$  ( $T$  is countable), then the embedding is given by  $\varepsilon \mapsto \sum_{s \leq \varepsilon} e_s$ .

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Laboratoire de Mathématiques, UMR 6623  
Université de Franche-Comté,

25030 Besançon, cedex - France

<sup>†</sup>florent.baudier@math.univ-fcomte.fr

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Aharoni proved in [1] that every separable metric space embeds into  $c_0$ , so  $T$  does.

The main result of this article is an improvement of Bourgain's metrical characterization of super-reflexivity. Bourgain proved in [2] that  $X$  is not super-reflexive if and only if the finite trees  $T_n$  uniformly embed into  $X$  (i.e with embedding constants independent of  $n$ ). Obviously if  $T$  embeds into  $X$  then the  $T'_n$ 's embed uniformly into  $X$  and  $X$  is not super-reflexive, but if  $X$  is not super-reflexive we did not know whether the infinite tree  $T$  embeds into  $X$ . In this paper, we prove that it is indeed the case :

**Theorem 1.1.** *Let  $X$  be a non super-reflexive Banach space, then  $(T, \rho)$  embeds into  $X$ .*

The proof of the direct part of Bourgain's Theorem essentially uses James' characterization of super-reflexivity (see [7]) and an enumeration of the finite trees  $T_n$ . We recall James' Theorem :

**Theorem 1.2** (James). *Let  $0 < \theta < 1$  and  $X$  a non super-reflexive Banach space, then :  $\forall n \in \mathbb{N}, \exists x_1, x_2, \dots, x_n \in B_X, \exists x_1^*, x_2^*, \dots, x_n^* \in B_{X^*}$  s.t :*

$$\begin{aligned} x_k^*(x_j) &= \theta \quad \forall k < j \\ x_k^*(x_j) &= 0 \quad \forall k \geq j \end{aligned}$$

## 2. METRICAL CHARACTERIZATION OF SUPER-REFLEXIVITY

The main obstruction to the embedding of  $T$  into any non-super-reflexive Banach space  $X$  is the finiteness of the sequences in James' characterization. How, with a sequence of Bourgain's type embedding, can we construct a single embedding from  $T$  into  $X$  ?

In [13], Ribe shows in particular, that  $\bigoplus_2 l_{p_n}$  and  $(\bigoplus_2 l_{p_n}) \oplus l_1$  are uniformly homeomorphic, where  $(p_n)_n$  is a sequence of numbers such that  $p_n > 1$ , and  $p_n$  tends to 1. But  $T$  embeds into  $l_1$ , hence via the uniform homeomorphism  $T$  embeds into  $\bigoplus_2 l_{p_n}$ . However  $T$  does not embed into any  $l_{p_n}$  (they are super-reflexive).

The problem solved in the next theorem, inspired in part by Ribe's proof, is to construct a subspace with a Schauder decomposition  $\bigoplus F_n$  where  $T_{2n+1}$  embeds into  $F_n$  and to repart properly the embeddings in order to obtain the desired embedding.

*Proof of Theorem 1.1 :* Let  $(\varepsilon_i)_{i \geq 0}$ , a sequence of positive real numbers such that  $\prod_{i \geq 0} (1 + \varepsilon_i) \leq 2$ , and fix  $0 < \theta < 1$ . Let  $k_n = 2^{2^{n+1}+1} - 1$ . First we construct inductively a sequence  $(F_n)_{n \geq 0}$  of subspaces of  $X$ , which is a Schauder finite dimensional decomposition of a subspace of  $X$  s.t the projection from  $\bigoplus_{i=0}^q F_i$  onto  $\bigoplus_{i=0}^p F_i$ , with kernel  $\bigoplus_{i=p+1}^q F_i$  (with  $p < q$ ) is of norm at most  $\prod_{i=p}^{q-1} (1 + \varepsilon_i)$ , and sequences

$$\begin{aligned} x_{n,1}, x_{n,2}, \dots, x_{n,k_n} &\in B_{F_n} \\ x_{n,1}^*, x_{n,2}^*, \dots, x_{n,k_n}^* &\in B_{X^*} \end{aligned}$$

s.t :

$$\begin{aligned} x_{n,k}^*(x_{n,j}) &= \theta \quad \forall k < j \\ x_{n,k}^*(x_{n,j}) &= 0 \quad \forall k \geq j. \end{aligned}$$

Denote  $\Phi_n : T_n \rightarrow \{1, 2, \dots, 2^{n+1} - 1\}$  the enumeration of  $T_n$  following the lexicographic order. It is an enumeration of  $T_n$  such that any pair of segments in  $T_n$  starting at incomparable nodes (with respect to the tree ordering  $\leq$ ) are mapped inside disjoint intervals.

Let  $\Psi_n = \Phi_{2^{n+1}}$  and  $\Gamma_n = T_{2^{n+1}}$ .

$X$  is non super-reflexive, hence from James' Theorem :  
 $\exists x_{0,1}, x_{0,2}, \dots, x_{0,7} \in B_X, \exists x_{0,1}^*, x_{0,2}^*, \dots, x_{0,7}^* \in B_{X^*}$  s.t :

$$\begin{aligned} x_{0,k}^*(x_{0,j}) &= \theta \quad \forall k < j \\ x_{0,k}^*(x_{0,j}) &= 0 \quad \forall k \geq j. \end{aligned}$$

$\Gamma_0 = T_2$  embeds into  $X$  via the embedding  $f_0(\varepsilon) = \sum_{s \leq \varepsilon} x_{0,\Psi_0(s)}$  (see [2]).  
Let  $F_0 = \text{Span}\{x_{0,1}, \dots, x_{0,7}\}$ , then  $\dim(F_0) < \infty$ .

Suppose that  $F_0, \dots, F_p$ , and

$$\begin{aligned} x_{p,1}, x_{p,2}, \dots, x_{p,k_p} &\in B_{F_p} \\ x_{p,1}^*, x_{p,2}^*, \dots, x_{p,k_p}^* &\in B_{X^*} \end{aligned}$$

verifying the required conditions, are constructed for all  $p \leq n$ .

We apply Mazur's Lemma (see [9] page 4) to the finite dimensional subspace  $\bigoplus_{i=0}^n F_i$  of  $X$ .  
Thus there exists  $Y_n \subset X$  such that  $\dim(X/Y_n) < \infty$  and :

$$\|x\| \leq (1 + \varepsilon_n) \|x + y\|, \forall (x, y) \in \bigoplus_{i=0}^n F_i \times Y_n$$

But  $Y_n$  is of finite codimension in  $X$ , hence is not super-reflexive.  
From James' Theorem and Hahn-Banach Theorem:

$$\begin{aligned} \exists x_{n+1,1}, x_{n+1,2}, \dots, x_{n+1,k_{n+1}} &\in B_{Y_n}, \\ \exists x_{n+1,1}^*, x_{n+1,2}^*, \dots, x_{n+1,k_{n+1}}^* &\in B_{X^*}, \end{aligned}$$

s.t :

$$\begin{aligned} x_{n+1,k}^*(x_{n+1,j}) &= \theta \quad \forall k < j \\ x_{n+1,k}^*(x_{n+1,j}) &= 0 \quad \forall k \geq j. \end{aligned}$$

$\Gamma_{n+1}$  embeds into  $Y_n$  via the embedding  $f_{n+1}(\varepsilon) = \sum_{s \leq \varepsilon} x_{n+1,\Psi_{n+1}(s)}$ .  
Let  $F_{n+1} = \text{Span}\{x_{n+1,j} ; 1 \leq j \leq k_{n+1}\}$ , then  $\dim(F_{n+1}) < \infty$ , which achieves the induction.

Now define the following projections :

Let,  $P_n$  the projection from  $\overline{\text{Span}}(\bigcup_{i=0}^n F_i)$  onto  $F_0 \bigoplus \dots \bigoplus F_n$  with kernel  $\overline{\text{Span}}(\bigcup_{i=n+1}^{\infty} F_i)$ .

It is easy to show that  $\|P_n\| \leq \prod_{i=n}^{\infty} (1 + \varepsilon_i) \leq 2$ .

We denote now  $\Pi_0 = P_0$  and  $\Pi_n = P_n - P_{n-1}$  for  $n \geq 1$ . We have that  $\|\Pi_n\| \leq 4$ .

From Bourgain's construction, for all  $n$  :

$$(1) \quad \frac{\theta}{3} \rho(\varepsilon, \varepsilon') \leq \|f_n(\varepsilon) - f_n(\varepsilon')\| \leq \rho(\varepsilon, \varepsilon'),$$

where  $f_n$  denotes the Bourgain's type embedding from  $\Gamma_n$  in  $F_n$ , i.e  $f_n(\varepsilon) = \sum_{s \leq \varepsilon} x_{n, \Psi_n(s)}$ .

Note that :

$$\forall n, \forall \varepsilon \in \Gamma_n \quad \|f_n(\varepsilon)\| \leq |\varepsilon|.$$

Now we define our embedding.

Let

$$f : \quad T \quad \rightarrow \quad Y = \overline{\text{Span}}(\bigcup_{i=0}^{\infty} F_i) \subset X$$

$$\varepsilon \quad \mapsto \quad \lambda f_n(\varepsilon) + (1 - \lambda) f_{n+1}(\varepsilon), \quad \text{if } 2^n \leq |\varepsilon| \leq 2^{n+1}$$

where,

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n}$$

And

$$f(\emptyset) = 0.$$

We will prove that :

$$(2) \quad \forall \varepsilon, \varepsilon' \in T \quad \frac{\theta}{24} \rho(\varepsilon, \varepsilon') \leq \|f(\varepsilon) - f(\varepsilon')\| \leq 9 \rho(\varepsilon, \varepsilon').$$

**Remark 2.1** We have  $\frac{\theta}{24} |\varepsilon| \leq \|f(\varepsilon)\| \leq |\varepsilon|$ .

First of all, we show that  $f$  is 9–Lipschitz.

We can suppose that  $0 < |\varepsilon| \leq |\varepsilon'|$  w.r.t remark 2.1.

If  $|\varepsilon| \leq \frac{1}{2} |\varepsilon'|$  then :

$$\rho(\varepsilon, \varepsilon') \geq |\varepsilon'| - |\varepsilon| \geq \frac{|\varepsilon| + |\varepsilon'|}{3}$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \leq 3 \rho(\varepsilon, \varepsilon').$$

If  $\frac{1}{2} |\varepsilon'| < |\varepsilon| \leq |\varepsilon'|$ , we have two different cases to consider.

1) if  $2^n \leq |\varepsilon| \leq |\varepsilon'| < 2^{n+1}$ .

Then, let

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n} \quad \text{and} \quad \lambda' = \frac{2^{n+1} - |\varepsilon'|}{2^n}.$$

$$\begin{aligned} \|f(\varepsilon) - f(\varepsilon')\| &= \|\lambda f_n(\varepsilon) - \lambda' f_n(\varepsilon') + (1 - \lambda) f_{n+1}(\varepsilon) - (1 - \lambda') f_{n+1}(\varepsilon')\| \\ &\leq \lambda \|f_n(\varepsilon) - f_n(\varepsilon')\| + |\lambda - \lambda'| (\|f_n(\varepsilon')\| + \|f_{n+1}(\varepsilon')\|) + (1 - \lambda) \|f_{n+1}(\varepsilon) - f_{n+1}(\varepsilon')\| \\ &\leq \rho(\varepsilon, \varepsilon') + 2\rho(\varepsilon, \varepsilon') + 2\rho(\varepsilon, \varepsilon') \\ &\leq 5\rho(\varepsilon, \varepsilon'), \end{aligned}$$

because  $\|f_n(\varepsilon')\| < 2^{n+1}$ ,  $\|f_{n+1}(\varepsilon')\| < 2^{n+1}$  and,

$$|\lambda - \lambda'| = \frac{|\varepsilon'| - |\varepsilon|}{2^n} \leq \frac{\rho(\varepsilon, \varepsilon')}{2^n}.$$

2) if  $2^n \leq |\varepsilon| \leq 2^{n+1} \leq |\varepsilon'| < 2^{n+2}$ .

Then, let

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n} \text{ and } \lambda' = \frac{2^{n+2} - |\varepsilon'|}{2^{n+1}}.$$

$$\begin{aligned} \|f(\varepsilon) - f(\varepsilon')\| &= \|\lambda f_n(\varepsilon) + (1 - \lambda)f_{n+1}(\varepsilon) - \lambda' f_{n+1}(\varepsilon') - (1 - \lambda')f_{n+2}(\varepsilon')\| \\ &\leq \lambda(\|f_n(\varepsilon)\| + \|f_{n+1}(\varepsilon)\|) + (1 - \lambda')(\|f_{n+1}(\varepsilon')\| + \|f_{n+2}(\varepsilon')\|) + \|f_{n+1}(\varepsilon) - f_{n+1}(\varepsilon')\| \\ &\leq \rho(\varepsilon, \varepsilon') + 2\lambda|\varepsilon| + 2(1 - \lambda')|\varepsilon'| \\ &\leq 9\rho(\varepsilon, \varepsilon'), \end{aligned}$$

because,

$$\lambda \leq \frac{\rho(\varepsilon, \varepsilon')}{2^n}, \text{ so } \lambda|\varepsilon| \leq 2\rho(\varepsilon, \varepsilon').$$

Similarly

$$1 - \lambda' = \frac{|\varepsilon'| - 2^{n+1}}{2^{n+1}} \leq \frac{\rho(\varepsilon, \varepsilon')}{2^{n+1}} \text{ and } (1 - \lambda')|\varepsilon'| \leq 2\rho(\varepsilon, \varepsilon').$$

Finally,  $f$  is 9-Lipschitz.

Now we deal with the minoration.

In our next discussion, whenever  $|\varepsilon|$  (respectively  $|\varepsilon'|$ ) will belong to  $[2^n, 2^{n+1})$ , for some integer  $n$ , we shall denote

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n} \text{ (respectively } \lambda' = \frac{2^{n+1} - |\varepsilon'|}{2^n}).$$

We can suppose that  $\varepsilon$  is smaller than  $\varepsilon'$  in the lexicographic order. Denote  $\delta$  the greatest common ancestor of  $\varepsilon$  and  $\varepsilon'$ . And let  $d = |\varepsilon| - |\delta|$  (respectively  $d' = |\varepsilon'| - |\delta|$ ).

1) if  $2^n \leq |\varepsilon|, |\varepsilon'| \leq 2^{n+1}$ .

We have,

$$x_{n, \Psi_n(\delta)}^* \Pi_n(f(\varepsilon) - f(\varepsilon')) = \theta(\lambda d - \lambda' d').$$

$$x_{n+1, \Psi_{n+1}(\delta)}^* \Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) = \theta((1 - \lambda)d - (1 - \lambda')d').$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \geq \frac{\theta(d - d')}{8}.$$

And,

$$-x_{n, \Psi_n(\varepsilon)}^* \Pi_n(f(\varepsilon) - f(\varepsilon')) = \theta\lambda' d'$$

$$-x_{n+1, \Psi_{n+1}(\varepsilon)}^* \Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) = \theta(1 - \lambda')d'.$$

So,

$$\|f(\varepsilon) - f(\varepsilon')\| \geq \frac{\theta d'}{8}.$$

Finally if we distinguish the cases  $\frac{d}{2} \leq d'$ , and  $d' < \frac{d}{2}$  we obtain :

$$\|f(\varepsilon) - f(\varepsilon')\| \geq \frac{\theta(d + d')}{24} = \frac{\theta}{24} \rho(\varepsilon, \varepsilon').$$

2) if  $2^n \leq |\varepsilon| \leq 2^{n+1} \leq 2^{q+1} \leq |\varepsilon'| \leq 2^{q+2}$ ,  
 or  $2^n \leq |\varepsilon'| \leq 2^{n+1} \leq 2^{q+1} \leq |\varepsilon| \leq 2^{q+2}$ .

If  $n < q$ ,

$$|x_{q+1, \Psi_{q+1}(\delta)}^* \Pi_{q+1}(f(\varepsilon) - f(\varepsilon')) + x_{q+2, \Psi_{q+2}(\delta)}^* \Pi_{q+2}(f(\varepsilon) - f(\varepsilon'))| = \theta \text{Max}(d, d')$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \geq \frac{\theta}{16} \rho(\varepsilon, \varepsilon').$$

If  $n = q$  and  $|\varepsilon| \leq |\varepsilon'|$ ,

$$|x_{n+1, \Psi_{n+1}(\varepsilon)}^* \Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) + x_{n+2, \Psi_{n+2}(\delta)}^* \Pi_{n+2}(f(\varepsilon) - f(\varepsilon'))| \geq \theta d'.$$

So,

$$\|f(\varepsilon) - f(\varepsilon')\| \geq \frac{\theta}{16} \rho(\varepsilon, \varepsilon').$$

If  $n = q$  and  $|\varepsilon'| < |\varepsilon|$ ,

$$x_{n+1, \Psi_{n+1}(\delta)}^* \Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) - x_{n+1, \Psi_{n+1}(\varepsilon)}^* \Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) + x_{n+2, \Psi_{n+2}(\delta)}^* \Pi_{n+2}(f(\varepsilon) - f(\varepsilon')) = \theta d.$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \geq \frac{\theta}{24} \rho(\varepsilon, \varepsilon').$$

Finally  $T \xrightarrow{\frac{216}{\theta}} X$ .

□

**Corollary 2.2.** *X is non super-reflexive if and only if  $(T, \rho)$  embeds into X.*

*Proof :* It follows clearly from Bourgain's result [2] and Theorem 1.1. □

## 3. METRIC CHARACTERIZATION OF THE LINEAR TYPE

First we identify canonically  $\{-1, 1\}^n$  with  $K_n = \{-1, 1\}^n \times \prod_{k>n} \{0\}$ .

Let  $p \in [1, \infty)$ .

Then we define an other metric on  $T = \bigcup K_n$  as follows :

$\forall \varepsilon, \varepsilon' \in T$ ,

$$d_p(\varepsilon, \varepsilon') = \left( \sum_{i=0}^{\infty} |\varepsilon_i - \varepsilon'_i|^p \right)^{\frac{1}{p}}.$$

The length of  $\varepsilon \in T$  can be viewed as  $|\varepsilon| = (d_p(\varepsilon, 0))^p$ .

The norm  $\|\cdot\|_p$  on  $\ell_p$  coincides with  $d_p$  for the elements of  $T$ .

We recall now two classical definitions :

Let  $X$  and  $Y$  be two Banach spaces. If  $X$  and  $Y$  are linearly isomorphic, the *Banach-Mazur distance* between  $X$  and  $Y$ , denoted by  $d_{BM}(X, Y)$ , is the infimum of  $\|T\| \|T^{-1}\|$ , over all linear isomorphisms  $T$  from  $X$  onto  $Y$ .

For  $p \in [1, \infty]$ , we say that a Banach space  $X$  uniformly contains the  $\ell_p^n$ 's if there is a constant  $C \geq 1$  such that for every integer  $n$ ,  $X$  admits an  $n$ -dimensional subspace  $Y$  so that  $d_{BM}(\ell_p^n, Y) \leq C$ .

We state and prove now the following result.

**Theorem 3.1.** *Let  $p \in [1, \infty)$ .*

*If  $X$  uniformly contains the  $\ell_p^n$ 's then  $(T, d_p)$  embeds into  $X$ .*

*Proof* : We first recall a fundamental result due to Krivine (for  $1 < p < \infty$  in [8]) and James (for  $p = 1$  and  $\infty$  in [7]).

**Theorem 3.2** (James-Krivine). *Let  $p \in [1, \infty]$  and  $X$  be a Banach space uniformly containing the  $\ell_p^n$ 's. Then, for any finite codimensional subspace  $Y$  of  $X$ , any  $\epsilon > 0$  and any  $n \in \mathbb{N}$ , there exists a subspace  $F$  of  $Y$  such that  $d_{BM}(\ell_p^n, F) < 1 + \epsilon$ .*

Using Theorem 3.2 together with the fact that each  $\ell_p^n$  is finite dimensional, we can build inductively finite dimensional subspaces  $(F_n)_{n=0}^{\infty}$  of  $X$  and  $(R_n)_{n=0}^{\infty}$  so that for every  $n \geq 0$ ,  $R_n$  is a linear isomorphism from  $\ell_p^n$  onto  $F_n$  satisfying

$$\forall u \in \ell_p^n \quad \frac{1}{2} \|u\| \leq \|R_n u\| \leq \|u\|$$

and also such that  $(F_n)_{n=0}^{\infty}$  is a Schauder finite dimensional decomposition of its closed linear span  $Z$ . More precisely, if  $P_n$  is the projection from  $Z$  onto  $F_0 \oplus \dots \oplus F_n$  with kernel  $\overline{\text{Span}(\bigcup_{i=n+1}^{\infty} F_i)}$ , we will assume as we may, that  $\|P_n\| \leq 2$ . We denote now  $\Pi_0 = P_0$  and  $\Pi_n = P_n - P_{n-1}$  for  $n \geq 1$ . We have that  $\|\Pi_n\| \leq 4$ .

We now consider  $\varphi_n : T_n \rightarrow \ell_p^n$  defined by

$$\forall \varepsilon \in T_n, \quad \varphi_n(\varepsilon) = \sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i,$$

where  $(e_i)$  is the canonical basis of  $\ell_p^n$ . The map  $\varphi_n$  is clearly an isometric embedding of  $T_n$  into  $\ell_p^n$ .

Then we set :

$$\forall \varepsilon \in T_n, \quad f_n(\varepsilon) = R_n(\varphi_n(\varepsilon)) \in F_n.$$

Finally we construct a map  $f : T \rightarrow X$  as follows :

$$f : \quad T \quad \rightarrow \quad X$$

$$\varepsilon \quad \mapsto \quad \lambda f_m(\varepsilon) + (1 - \lambda) f_{m+1}(\varepsilon), \quad \text{if } 2^m \leq |\varepsilon| < 2^{m+1},$$

where,

$$\lambda = \frac{2^{m+1} - |\varepsilon|}{2^m}.$$

**Remark 3.3** We have  $\frac{1}{16}|\varepsilon|^{\frac{1}{p}} \leq \|f(\varepsilon)\| \leq |\varepsilon|^{\frac{1}{p}}$ .

Like in the proof of Theorem 1.1, we prove that  $f$  is 9-Lipschitz using exactly the same computations.

We shall now prove that  $f^{-1}$  is Lipschitz. We consider  $\varepsilon, \varepsilon' \in T$  and assume again that  $0 < |\varepsilon| \leq |\varepsilon'|$ . We need to study two different cases. Again, whenever  $|\varepsilon|$  (respectively  $|\varepsilon'|$ ) will belong to  $[2^m, 2^{m+1})$ , for some integer  $m$ , we shall denote

$$\lambda = \frac{2^{m+1} - |\varepsilon|}{2^m} \quad (\text{respectively } \lambda' = \frac{2^{m+1} - |\varepsilon'|}{2^m}).$$

1) if  $2^m \leq |\varepsilon|, |\varepsilon'| < 2^{m+1}$ .

$$\begin{aligned} d_p(\varepsilon, \varepsilon') &\leq \|\lambda \sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i - \lambda' \sum_{i=1}^{|\varepsilon'|} \varepsilon'_i e_i\|_p + \|(1 - \lambda) \sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i - (1 - \lambda') \sum_{i=1}^{|\varepsilon'|} \varepsilon'_i e_i\|_p \\ &\leq 2\|\Pi_m(f(\varepsilon) - f(\varepsilon'))\| + 2\|\Pi_{m+1}(f(\varepsilon) - f(\varepsilon'))\| \\ &\leq 16\|f(\varepsilon) - f(\varepsilon')\|. \end{aligned}$$

2) if  $2^m \leq |\varepsilon| \leq 2^{m+1} \leq 2^{q+1} \leq |\varepsilon'| < 2^{q+2}$ .

if  $m < q$ ,

$$\begin{aligned} d_p(\varepsilon, \varepsilon') &\leq 2d_p(\varepsilon', 0) \\ &\leq 2((1 - \lambda')d_p(\varepsilon', 0) + \lambda'd_p(\varepsilon', 0)) \\ &\leq 2(2\|\Pi_{q+2}(f(\varepsilon) - f(\varepsilon'))\| + 2\|\Pi_{m+1}(f(\varepsilon) - f(\varepsilon'))\|) \\ &\leq 32\|f(\varepsilon) - f(\varepsilon')\|. \end{aligned}$$

if  $m = q$ ,

$$\begin{aligned}
d_p(\varepsilon, \varepsilon') &\leq \lambda d_p(\varepsilon, 0) + \|(1 - \lambda) \sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i - \lambda' \sum_{i=1}^{|\varepsilon'|} \varepsilon'_i e_i\|_p + (1 - \lambda') d_p(\varepsilon', 0) \\
&\leq 2\|\Pi_m(f(\varepsilon) - f(\varepsilon'))\| + 2\|\Pi_{m+1}(f(\varepsilon) - f(\varepsilon'))\| + 2\|\Pi_{m+2}(f(\varepsilon) - f(\varepsilon'))\| \\
&\leq 24\|f(\varepsilon) - f(\varepsilon')\|.
\end{aligned}$$

Finally we obtain that  $f^{-1}$  is 32-Lipschitz, and  $T \xrightarrow{288} X$ .

□

In the sequel a Banach space  $X$  is said to have *type*  $p > 0$  if there exists a constant  $T < \infty$  such that for every  $n$  and every  $x_1, \dots, x_n \in X$ ,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^p \leq T^p \sum_{j=1}^n \|x_j\|_X^p,$$

where the expectation  $\mathbb{E}_\varepsilon$  is with respect to a uniform choice of signs  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}^n$ .

The set of  $p$ 's for which  $X$  contains  $\ell_p^n$ 's uniformly is closely related to the type of  $X$  according to the following result due to Maurey, Pisier [10] and Krivine [8], which clarifies the meaning of these notions.

**Theorem 3.4** (Maurey-Pisier-Krivine). *Let  $X$  be an infinite-dimensional Banach space. Let*

$$p_X = \sup\{p ; X \text{ is of type } p\},$$

*Then  $X$  contains  $\ell_p^n$ 's uniformly for  $p = p_X$ . Equivalently, we have*

$$p_X = \inf\{p ; X \text{ contains } \ell_p^n \text{ uniformly}\}.$$

We deduce from Theorem 3.1 two corollaries.

**Corollary 3.5.** *Let  $X$  a Banach space and  $1 \leq p < 2$ . The following assertions are equivalent :*

- i)  $p_X \leq p$ .
- ii)  $X$  uniformly contains the  $\ell_p^n$ 's.
- iii) the  $(T_n, d_p)$ 's uniformly embed into  $X$ .
- iv)  $(T, d_p)$  embeds into  $X$ .

*Proof :* ii) implies i) is obvious.

i) implies ii) is due to Theorem 3.2 and the work of Bretagnolle, Dacunha-Castelle and Krivine [4].

For the equivalence between ii) and iii) see the work of Bourgain, Milman and Wolfson [3] and Krivine [8].

iv) implies iii) is obvious.

And ii) implies iv) is Theorem 3.1.

□

**Corollary 3.6.** *Let  $X$  be an infinite dimensional Banach space, then  $(T, d_2)$  embeds into  $X$ .*

*Proof :* This corollary is a consequence of the Dvoretzky's Theorem [6] and Theorem 3.1.

□

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